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A SUPPLEMENT TO LAWDEN'S THEORY*

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An extension of the mathematical model of the motion of a particle of variable mass in a central gravitational field is proposed, based on a discrete flow of the reactive mass and jump-type variation of the direction of the reactive force. The problem of programming the optimal orbital transitions is studied, in the case when, as distinct from /1, 2/, the transit time is fixed. As a result, the possible pieces of optimal transitions, corresponding to impulsive, zero, and intermediate thrust, are described. It is shown that intermediate thrust generates motion along spirals which are not the same as Lawden's spiral.

1. Generalization of the equations of motion of a particle of variable mass in a central gravitational field. We know that the analogue of Meshcherskii's equation in the case of the plane motion of a particle of variable mass in a central gravitational field is

$$r^{\cdot\cdot} = f(r, \chi) + m^{-1}P\cos\theta, \quad f = -\nu r^{-2} + \chi^2 r^{-3} \tag{1.1}$$

$$\psi^{\cdot} = r^{-2}\chi, \ \chi^{\cdot} = rm^{-1}P\sin\theta, \ m^{\cdot} = -c^{-1}P$$

Here, r, ψ are the particle polar coordinates, χ is the sectoral velocity, ν is the gravitational constant, m is the mass of the particle, c is the specific impulse of the thrust P, and the angle θ characterizes the direction of the reactive force (Fig.1).

In the classical sense the operations of differentiation of Eqs.(1.1) are only meaningful for ordinary /3/ (e.g., piecewise continuous) programs $P(\cdot)$, $\theta(\cdot)$ However, some problems

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concerning optimal interorbital transitions, have no solutions in the class of ordinary programs $P(\cdot)$. Their approximation by solvable problems shows /1/ that the approximate optimal programs $P(\cdot)$ have a tendency to reproduce the Dirac δ-impulse..

The assumption of impulsive thrust is only possible when the differentiation in the last of Eqs.(1.1) is interpreted in the sense of generalized function theory /3/. The result is that then, corresponding to the impulsive thrust, we have jump-type variation of the mass. This leads at once to the problem of multiplying the discontinuous function m^{-1} by the impulsive thrust, and then, possibly, by the discontinuous program $\theta(\cdot)$ Study of this problem has shown independent smooth approximation of the impulse, and the jump in its direction can lead to different trajectories. The reason for this indeterminacy is that system (1.1) does not have the Frobenius property /4, 5/. Thus, given independent choice of the programs for varying the impulsive thrust and its direction, we cannot expect the trajectory generated by them to be unique. Our device below for matching the program $P(\cdot), \theta(\cdot)$ leads to a unique and stable trajectory.

We regard the pair $\{P(\cdot), \theta(\cdot)\}$ as matched if the thrust $P = -cDm$, running mass varies piecewise continuously, and for some piecewise continuous function $\sigma(\cdot)$, the angle $\theta(t) = \sigma(s(t))$, where $s = c \ln m$. Here, D denotes generalized differentiation.

Note that a pair $\{P(\cdot), \theta(\cdot)\}$ with piecewise continuous components is equivalent to a matched pair. It differs from the initial pair only in that, on the pieces where the thrust is zero, the direction of the thrust is constant.

A matched pair is uniquely determined by the pair of programs $\{s(\cdot), \sigma(\cdot)\}$:

$$P = -cDm, \; m = \exp(c^{-1}s), \; \theta = \sigma(s) \tag{1.2}$$

It can be shown that the equations of motion, generated by a matched pair, of a particle of variable mass in a central gravitational field, are

$$\begin{gathered} r^{\cdot} = y_2 + w_2(s); \; y_2^{\cdot} = f(r, \chi), \; \chi = y_4 + rw_4(s) \\ \psi^{\cdot} = r^{-2}\chi; \; y_4^{\cdot} = -r^{\cdot}w_4(s) \\ w_2(s) = \int_s^{s(0)} \cos\sigma(\xi)\,d\xi, \quad w_4(s) = \int_s^{s(0)} \sin\sigma(\xi)\,d\xi \\ y_2(0) = r^{\cdot}(0), \; y_4(0) = \chi(0) \end{gathered} \tag{1.3}$$

Here, the y_i play an auxiliary role.

Given the solution of system (1.3), we can indicate the parameters of the particle motion

$$r, r^{\cdot} = y_2 + w_2(s), \psi, \chi = y_4 + rw_4(s) \tag{1.4}$$

that correspond to programs (1.2) of variation of the thrust and its direction.

Note the following:

If the thrust varies piecewise-continuously, then system (1.1), (1.2) is equivalent to system (1.3).

If the sequences of smooth programs $\{s_k(\cdot)\}, \{\sigma_k(\cdot)\}$ are convergent respectively to the programs $s(\cdot), \sigma(\cdot)$ at their points of continuity, then the sequence of trajectories $(s = s_k, \sigma = \sigma_k)$ of system (1.1), (1.2) is convergent to the trajectory (1.4) at its points of continuity.

System (1.3) is equivalent to (1.1), (1.2) provided that the differentiation in the latter is generalized and we use the following rule for multiplying discontinuous by impulsive functions:

$$cm^{-1}P\cos\sigma(s) = -Dw_2(s), \; cm^{-1}P\sin\sigma(s) = -Dw_4(s)$$

These relations hold in the usual sense for piecewise continuous thrust. Their right-hand sides are meaningful for impulsive thrust, and serve as a definition for the left-hand sides, which then become meaningless.

Let us write the expressions for the increments of the radial and sectoral velocities due to jump-type variation of the mass:

$$\Delta r^{\cdot} = -\int_{s(t-0)}^{s(t+0)} \cos\sigma(\xi)\,d\xi, \quad \Delta\chi = -r(t)\int_{s(t-0)}^{s(t+0)} \sin\sigma(\xi)\,d\xi$$

$$\Delta s = s(t+0) - s(t-0) = c\ln(m(t+0)(m(t-0))^{-1})$$

Let the function $\sigma(s)$ be constant in the interval $[s(t-0), s(t+0)]$. We then have an analogue of Tsiolkovskii's formula:

$$\Delta r^{\cdot} = -\Delta s(t)\cos\sigma(s(t)), \quad \Delta\chi = -\Delta s(t)\,r(t)\sin\sigma(s(t)) \tag{1.5}$$

Consequently, our approach corresponds to the treatment of /6, p.86/ of "the instantaneous ejection of a finite mass seen as the idealization of a continuous ejection in an infinitely short time".

2. The problem of optimizing interorbital flight in a fixed time, its formalization and reduction. We consider the problem of dynamic optimization (/7/, p.86) (t_p is a fixed instant)

$$r(t_p) \to \max \tag{2.1}$$

under the dynamic relations (1.1). We assume that, at the start of the manoeuvre, $r(0) = r_0$, $r^{\cdot}(0) = r_0^{\cdot}$, $\chi(0) = \chi_0$, $m(0) = m_0$. At the final instant we must have

$$r^{\cdot}(t_p) = r_p^{\cdot},\ \chi(t_p) = \sqrt{\nu r(t_p)} + \Delta\chi_p,\ m(t_p) = m_p \tag{2.2}$$

If $r_p^{\cdot} = 0$, $\Delta\chi_p = 0$, we are speaking of flight in a circular orbit of maximum radius.

Let us formalize the problem. The optimal programs of variation of the thrust and its direction will be sought among the matched pairs. In this connection, we extend the differential relations (1.1), (1.2) to relations (1.3).

The problem is now stated as follows. We need to obtain condition (2.1) in the class of peicewise-continuous non-increasing functions $s(\cdot)$ such that $s(0) = c\ln m_0$, $s(t_p) = c\ln m_p$, and in the class of piecewise-continuous functions $\sigma(\cdot)$, defined in the interval $[s(t_p), s(0)]$.

3. The necessary conditions for optimality. To solve our problem we use Lagrange's principle. We note that the third of Eqs.(1.3) can be ignored, since the particle angular position does not appear in the other equations, in the minimized functional, or in the boundary conditions (2.2). We thus form the Lagrange functional

$$L = -r(t_p) + \mu_2 r^{\cdot}(t_p) + \mu_4(\chi(t_p) - \sqrt{\nu r(t_p)}) + \int_0^{t_p} (\lambda_1(y_2 + w_2 - r^{\cdot}) + \lambda_2(f - y_2^{\cdot}) - \lambda_4(r^{\cdot} w_4 + y_4^{\cdot}))\,dt \tag{3.1}$$

We next use integration by parts, introduce the Hamiltonian

$$H = (\lambda_1 - \lambda_4 w_4)\,r^{\cdot} + \lambda_2 f \tag{3.2}$$

use the last two relations and (1.3) to evaluate the variation δL, and as usual, take as the Lagrange multipliers λ_i the solution of the Cauchy problem for the conjugate system

$$\begin{gathered}\lambda_1^{\cdot} = -\partial H/\partial r,\ \lambda_1(t_p) = -1 + \mu_4(w_4(s(t_p)) - {}^1/_2\sqrt{\nu (r(t_p))^{-1}})\\ \lambda_2^{\cdot} = -\partial H/\partial r^{\cdot},\ \lambda_2(t_p) = \mu_2,\ \lambda_4^{\cdot} = -\partial H/\partial\chi,\ \lambda_4(t_p) = \mu_4\end{gathered} \tag{3.3}$$

Hence we have

$$\partial H / \partial w_2 = -\lambda_2^{\cdot},\ \partial H / \partial w_4 = -(\lambda_4 r)^{\cdot} \tag{3.4}$$

As a result, we obtain for the variation δL:

$$\begin{gathered}\delta L = \int_{s(t_p)}^{s(0)} ((\lambda_4 r)(t_p)\cos\sigma(\xi) - \lambda_2(t_p)\sin\sigma(\xi))\,\delta\sigma(\xi)\,d\xi + \int_0^{t_p}\int_{s(t_p)}^{s(0)} (\lambda_2^{\cdot}\sin\sigma(\xi) - \\ (\lambda_4 r)^{\cdot}\cos\sigma(\xi))\,\delta\sigma(\xi)\,d\xi\,dt + \int_0^{t_p} (\lambda_2^{\cdot}\cos\sigma(s) + (\lambda_4 r)^{\cdot}\sin\sigma(s))\,\delta s(t)\,dt\end{gathered} \tag{3.5}$$

Recall that an admissible program $s(\cdot)$ is non-increasing. Let $0 = t_0, t_1, \ldots, t_k = t_p$ be a non-decreasing sequence of instants, at which the function $s(\cdot)$ has discontinuities. In the intervals (t_i, t_{i+1}) the function $s(\cdot)$ is continuous and not increasing. Its inverse $s^{-1}(\cdot)$ is likewise not increasing, but may have discontinuities, corresponding to pieces of zero thrust. We complete its definition at points of discontinuity by continuity on the left. In these circumstances we can change the order of integration in the second integral in (3.5). The result is

$$\begin{gathered}\delta L = \sum_{i=0}^{k}\int_{s(t_i-0)}^{s(t_i+0)} (\lambda_2(t_i)\sin\sigma(\xi) - (\lambda_4 r)(t_i)\cos\sigma(\xi))\,\delta\sigma(\xi)\,d\xi + \\ \sum_{i=0}^{k-1}\int_{s(t_i-0)}^{s(t_i+0)} (-\lambda_2(s^{-1}(\xi))\sin\sigma(\xi) + (\lambda_4 r)(s^{-1}(\xi))\cos\sigma(\xi))\,\delta\sigma(\xi)\,d\xi +\end{gathered} \tag{3.6}$$

$$\int_0^{t_p} (\lambda_2^{\cdot} \cos\sigma(s(t)) + (\lambda_4 r)^{\cdot} \sin\sigma(s(t)))\, \delta s(t)\, dt$$

We will now analyse the variation (3.6), which must be non-negative for admissible variations $\delta s(\cdot), \delta\sigma(\cdot)$. It is natural to take the cases of impulsive, intermediate, and zero thrust.

Impulsive thrust. In this case we must have

$$\lambda_2(t_i) \sin\sigma(\xi) + (\lambda_4 r)(t_i) \cos\sigma(\xi) = 0,\ s(t_i + 0) < \xi < s(t_i - 0) \tag{3.7}$$

The following main conclusion can be drawn from this: at the instants of discrete flow of the reactive mass, provided that $\lambda_2(t_i)\lambda_4(t_i) \neq 0$, the line of action of the thrust remains unchanged. The direction of the thrust may reverse any finite number of times. In the absence of a selfcompensation effect, we have by (1.5) for the increments of radial and sectoral velocity

$$\Delta r^{\cdot}(t_i) = -\Delta s(t_i)\cos\theta(t_i),\ \Delta\chi(t_i) = -\Delta s(t_i) r(t_i) \sin\theta(t_i) \tag{3.8}$$

It is assumed here that $\theta(t_i) = \sigma(s(t_i + 0) + 0)$.

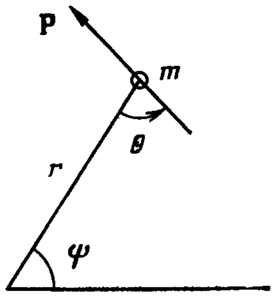

Fig.1

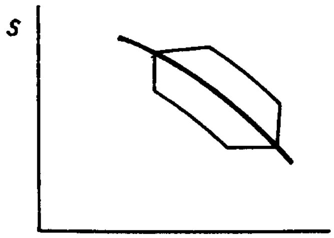

Fig.2

Intermediate thrust. This case corresponds to pieces of strict decrease of $s(\cdot)$. Analysis of the second term of (3.6) leads to the necessary condition

$$\lambda_2(t) \sin\sigma(s(t)) - (\lambda_4 r)(t) \cos\sigma(s(t)) = 0 \tag{3.9}$$

We turn to the third integral term of (3.6). Using the needle-type variations shown in Fig.2, we can conclude that

$$\lambda_2^{\cdot} \cos\sigma(s) + (\lambda_4 r)^{\cdot} \sin\sigma(s) = 0. \tag{3.10}$$

Eqs.(3.9) and (3.10) form a linear system in $\cos\sigma(s), \sin\sigma(s)$. Its determinant is therefore zero. Successive integration gives the integral of system (1.3), (3.3)

$$\lambda_2^2 + (\lambda_4 r)^2 = \text{const} \tag{3.11}$$

Differentiation of (3.11) leads to the identities

$$Q(s, t) = \lambda_2\alpha - \lambda_4 r\beta = 0 \tag{3.12}$$

$$\alpha = \lambda_1 + \lambda_4 r^{-1} y_4,\ \beta = \lambda_4 r^{\cdot} - \lambda_2 r^{-2}\chi \tag{3.13}$$

From (3.12), using (3.9), we obtain the identities

$$\partial Q/\partial t = 2\nu r^{-3}\lambda_2^2 - \nu r^{-1}\lambda_4^2 + \alpha^2 + \beta^2 = 0 \tag{3.14}$$

$$\partial^2 Q/\partial t^2 = -9\nu r^{-3}\lambda_2\alpha - 6\nu r^{-4} r^{\cdot}\lambda_2^2 + 9\nu r^{-4}\chi\lambda_2\lambda_4 = 0 \tag{3.15}$$

The last relation is equivalent to

$$\lambda_2(3r\alpha + 2r^{\cdot}\lambda_2 - 3\lambda_4\chi) = 0 \tag{3.16}$$

If the second factor on the left-hand side of (3.16) vanishes in a time interval, then, recalling the definition (3.13), the second of Eqs.(3.3) can be written as

$$\lambda_2^{\cdot} = {}^2/_3 r^{-1} r^{\cdot} \lambda_2 \tag{3.17}$$

Solving this, we have

$$\lambda_2 = \lambda_2(t_*) r^{-2/3}(t_*) r^{2/3},\ 0 < t_* < t_p \tag{3.18}$$

We can now claim that, in the case of intermediate thrust, the factor λ_2 has no zeros. For, the presence of such a zero would imply, by (3.18), that λ_2 is identically zero. Hence, by (3.9), we should have the identity $\lambda_4 \cos\theta(t) = 0$, and also, by (3.12), $\lambda_4\beta = 0$ i.e., $\lambda_4 r = 0$. By (3.14) we have $\nu r^{-1}\lambda_4^2 = \alpha^2$ Thus, if $\lambda_4 = 0$, then also $\lambda_1 = 0$, which contradicts the boundary condition $\lambda_1(t_p) = -1$ Hence $\lambda_4 \neq 0$, so that $\cos\theta(t) = 0$, $r^{\cdot} = 0$ This obviously cannot correspond to a piece with non-zero thrust.

Note that, in the present case, $\sin\theta(t) \neq 0$ Otherwise, by (3.9), $\lambda_4 = 0$, and by (3.12), $\alpha = \lambda_1 = 0$ But then, by (3.14), $(2\nu r^{-1} + r^{-4}\chi^2)\lambda_2^2 = 0$, i.e., $\lambda_2 = 0$, which contradicts our above conclusion.

Put $\varkappa = \operatorname{ctg}\theta$. By (3.9), $\varkappa = \lambda_2(\lambda_4 r)^{-1}$ Hence the derivative $\varkappa^{\cdot} = \lambda_2^{-1}\lambda_2^{\cdot}\varkappa - (\lambda_4 r)^{\cdot}\lambda_2^{-1}\varkappa^2$ Using (3.10) and (3.17) on the right-hand side of this expression, we obtain

$$\varkappa^{\cdot} = {}^2/_3 r^{-1} r^{\cdot} \varkappa (1 + \varkappa^2) \tag{3.19}$$

Like (3.17), this can be solved in quadratures:

$$\varkappa^{-2} + 1 = c_1 r^{-4/3}, \quad c_1 = (\varkappa^{-2}(t_*) + 1)\, r^{4/3}(t_*) \tag{3.20}$$

We have thereby obtained the optimal slope of this line of intermediate thrust. Let us now find the value of the thrust. It is found from the condition that the particle move over the so-called singular surface, given by (3.14), (3.15). But we shall first write its equations in terms of $\varkappa$. Note that (see (3.13))

$$\beta = r^{-1}\lambda_2 (\varkappa^{-1} r^{\cdot} - r^{-1}\chi) \tag{3.21}$$

Using (3.12) and (3.21), we transform (3.14) to

$$r(1 + \varkappa^2)(r^{\cdot} - \varkappa r^{-1}\chi)^2 = \nu\varkappa^2(1 - 2\varkappa^2) \tag{3.22}$$

Incidentally, it follows at once from this that $\varkappa$ is bounded:

$$|\varkappa| \leqslant 1/\sqrt{2} \tag{3.23}$$

Further, by (3.13), (3.12) and (3.21), we can obtain

$$\alpha = \varkappa^{-1}\beta = r^{-1}(\varkappa^{-1} r^{\cdot} - r^{-1}\chi)\varkappa^{-1}\lambda_2$$

Employing this expression in (3.16), we arrive at

$$r^{\cdot} = 6\varkappa(3 + 2\varkappa^2)^{-1}\chi r^{-1} \tag{3.24}$$

We can use this equation to eliminate the radial velocity from (3.22), with the result that we obtain for the sectoral velocity on the singular surface:

$$\chi = \sqrt{\nu r}\, g(\varkappa), \quad g(\varkappa) = \frac{3 + 2\varkappa^2}{3 - 2\varkappa^2}\sqrt{\frac{1 - 2\varkappa^2}{1 + \varkappa^2}} \tag{3.25}$$

It can be shown that motion over the singular surface is achieved under the action of the intermediate thrust

$$P = 3m\nu r^{-2}\varkappa \frac{9 - 22\varkappa^2 - 36\varkappa^4 - 40\varkappa^6}{(3 - 2\varkappa^2)^3(1 + \varkappa^2)^{1/2}} \operatorname{sign}(\sin\theta(t)) \tag{3.26}$$

Let us find the equation for the analogue of the Lawden spiral. The system consisting of the third equation of (1.3) and Eq.(3.24), has the integral (recalling (3.25))

$$\operatorname{arctg}\varkappa^{-1} - 3\varkappa^{-1} = 4c_1\psi + c_2 \tag{3.27}$$

where c_2 is an arbitrary constant, and the constant c_1 is given in (3.20).

All in all, we have the system of Eqs.(3.20), (3.27), which describe in parametric form a spiral which differs from Lawden's /1/. Since the parameter $\varkappa$ does not go beyond the bound (3.23), our spiral cannot, like Lawden's, turn around the centre of the gravitational field to infinity (see Fig.3, where the unshaded point corresponds to $\varkappa = 1/\sqrt{2}$).

Zero thrust in the interval (t_*, t_{**}). By the definition of an admissible program of variation of the thrust line in this piece, $\sigma(s(t)) = \theta_* = \text{const}$. The variation (see (3.6))

$$\delta L = \int_{t_*}^{t_{**}} (\lambda_2^{\cdot}\cos\theta_* + (\lambda_4 r)^{\cdot}\sin\theta_*)\,\delta s(t)\,dt$$

must be non-negative for any admissible variation δs, and in particular, for the variations shown in Fig.4. For the left-hand variation, using integration by parts, we have

$$\delta L = \varepsilon((\lambda_2(\tau) - \lambda_2(t_*))\cos\theta_* + ((\lambda_4 r)(\tau) - (\lambda_4 r)(t_*))\sin\theta_*) \geqslant 0$$

Similarly, using the right-hand variation, we can prove an inequality which is obtained from the last by replacing t_* by t_{**}. In all, we obtain the necessary conditions

$$\lambda_2(t)\cos\theta_* + (\lambda_4 r)(t)\sin\theta_* \geqslant \lambda_2(t_*)\cos\theta_* + (\lambda_4 r)(t_*)\sin\theta_* = \lambda_2(t_{**})\cos\theta_* + (\lambda_4 r)(t_{**})\sin\theta_{**}, \quad t_* \leqslant t \leqslant t_{**} \tag{3.28}$$

From the condition tht the length of the piece of zero thrust be independent, we can obtain, using (3.28),

$$\lambda_2(t_*) = \lambda_2(t_{**}), \ (r\lambda_4)(t_*) = (r\lambda_4)(t_{**}) \tag{3.29}$$

Relations (3.29) imply in particular that the optimal direction of thrust at the start and end of the piece is the same. It can be assumed that the direction remains the same in the interval (t_*, t_{**}).

Let t_i be an instant of impulsive thrust. The increment of the Hamiltonian can be found from the relation

$$H(t_i + 0) - H(t_i - 0) = \Delta s(t_i)(\lambda_2^{\cdot}(t_i \pm 0)\cos\theta(t_i) + (\lambda_4 r)^{\cdot}(t_i \pm 0)\sin\theta(t_i))$$

We can now conclude from the necessary conditions (3.28) that the Hamiltonian remains constant on the optimal trajectory and can only have a discontinuity at the first and last instants of the control process.

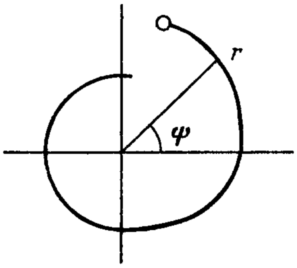

Fig.3

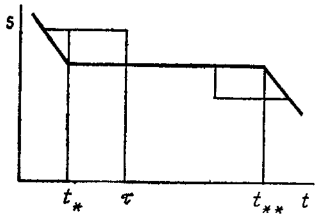

Fig.4

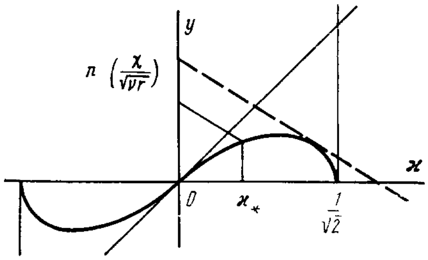

Fig.5

The conjugate system (3.3) can be integrated in quadratures (c_i are arbitrary constants)

$$\lambda_2 = c_1 r^{\cdot} + c_2 a^{-1}(3tr^{\cdot} - 2r + \omega r^2 f)$$
$$r\lambda_4 = \chi(rr^{\cdot})^{-1}\lambda_2 + c_2\chi^{-1}(rr^{\cdot})^{-1}\omega r^4 f + c_3 r$$
$$\lambda_1 = \lambda_4 w_4 + (c_2 - \lambda_2 f)(r^{\cdot})^{-1}, \ \omega = \chi^2(a\chi^2 + \nu^2)^{-1}$$
$$(rr^{\cdot})^2 = ar^2 + 2\nu r - \chi^2$$

Zone of impulsive shedding on the singular surface. Suppose we are given the set $(r, r^{\cdot}, \chi)$ of the object motion parameters. It is required to find the direction $\varkappa = \operatorname{ctg}\theta$ and the discrete mass flow Δm for which the set $(r, r^{\cdot} - \Delta s\cos\theta, \chi - \Delta sr\sin\theta)$ satisfies Eqs.(3.24), (3.25). We now obtain

$$\Delta s = (r\sin\theta_*)^{-1}(\chi - g(\varkappa_*)\sqrt{\nu r})$$

where $\varkappa_*$ is the root of the equation (see Fig.5)

$$r^{\cdot}\sqrt{\nu^{-1}r} - \chi\sqrt{(\nu r)^{-1}}\,\varkappa = \varkappa\sqrt{\frac{1-2\varkappa^2}{1+\varkappa^2}} = \rho(\varkappa)$$

Shedding is possible if $|r^{\cdot}\sqrt{\nu^{-1}r}| \leqslant n(\chi\sqrt{(\nu r)^{-1}})$, where n is found from the condition that the line $y = n - \varkappa\chi\sqrt{(\nu r)^{-1}}$ (shown broken in Fig.5) touches the curve $y = \rho(\varkappa)$ (the heavy line in Fig.5).

4. The case of a circular initial and final orbit. We will evaluate the Hamiltonian on the singular surface. By Eq.(3.17) and the first of (3.25), we have

$$H = \lambda_2(f - {}^2/_3 r^{-1}(r^{\cdot})^2),\quad f = \nu r^{-2}(g^2 - 1) < 0$$

On the singular surface, therefore, the Hamiltonian does not vanish. If the control process duration is not fixed, we can add to the conclusion of Sect.3 the fact that the Hamiltonian is continuous at the instants $0, t_p$ Hence the optimal control program does not contain intermediate thrust. It consists of apsidal tangential impulses. If the interorbital transition time is fixed, our analysis shows that the hypothesis of /2/ about the absence of intermediate thrust in problems with variable angular range is equivalent for our present case to continuity of the Hamiltonian at the instant of reaching the given orbit.

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ANALYTIC SOLUTIONS OF THE HAMILTON-JACOBI EQUATION OF AN IRREVERSIBLE SYSTEM IN THE NEIGHBOURHOOD OF A NON-DEGENERATE POTENTIAL ENERGY MAXIMUM*

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The existence of analytic solutions for the Hamilton-Jacobi equations of an irreversible system with two degrees of freedom in the neighbourhood of a non-degenerate maximum of potential energy is investigated. It is shown that these solutions define manifolds in phase space which are filled with trajectories which asymptotically approach an equilibrium position as $t \to \pm\infty$.

Consider a mechanical system with Lagrangian

$$L: R^2\{x\} \times R^2\{x^{\cdot}\} \to R,\quad L = T_2 + T_1 - \Pi$$
$$T_2 = {}^1/_2\langle K(x)\,x^{\cdot}, x^{\cdot}\rangle,\quad T_1 = \langle V(x), x^{\cdot}\rangle$$

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